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VOID NUCLEATION AND GROWTH FOR A CLASS OF INCOMPRESSIBLE NONLINEARLY ELASTIC MATERIALS

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Abstract—In this paper, a bifurcation problem for a solid sphere subjected to uniform tensile dead-loading p_0 at its boundary is examined within the framework of finite elastostatics. The sphere is composed of a particular class of homogeneous isotropic incompressible nonlinearly elastic materials, namely those of power-law type. One solution to the problem, for all values of p_0, corresponds to a homogeneous state in which the sphere remains undeformed while stressed. However, for sufficiently large values of p_0, there is in addition a second possible configuration involving an internal traction-free spherical cavity. The dependence on constitutive parameters of the critical load at which bifurcation occurs is examined as well as the subsequent void growth. The stress distribution after cavitation occurs is also described. The results are obtained in closed analytic form.

1. INTRODUCTION

Void nucleation and growth in solids have been of concern for a long time because of the fundamental role such phenomena play in fracture and other failure mechanisms. (See e.g. Goods and Brown, 1979 for a discussion of cavity nucleation in metals). The phenomenon of sudden void formation ("cavitation") has also been observed experimentally in vulcanized rubber by Gent and Lindley (1958). See also Williams and Schapery (1965). Nonlinear theories of solid mechanics have been used recently to account for such phenomena. The impetus for much of the recent theoretical developments have been supplied by the work of Ball (1982). Ball has made an extensive study of a class of *bifurcation problems* for the equations of nonlinear elasticity which model the appearance of a cavity in the interior of an apparently solid homogeneous isotropic elastic body once a critical load has been attained. An alternative interpretation for such problems in terms of the growth of a *pre-existing* micro-void has been given by Horgan and Abeyaratne (1986). Further investigations of such bifurcation problems have been carried out by Stuart (1985), Podio-Guidugli *et al.* (1986), Sivaloganathan (1986a,b), Chung *et al.* (1987), Antman and Negrón-Marrero (1987), Pericak-Spector and Spector (1988) and Horgan and Pence (1989a,b,c). It is worth noting that cavitation can be shown to occur only when *finite* strain measures are taken into account (see e.g. Horgan and Abeyaratne, 1986; Chung *et al.*, 1987). The corresponding problems in linearized elasticity or in the infinitesimal strain theory of plasticity do *not* exhibit such bifurcations.

The purpose of the present paper is to further investigate this bifurcation approach to void nucleation. We carry out an investigation of the problem of static tensile dead-loading of a solid sphere composed of a particular class of homogeneous isotropic incompressible nonlinearly elastic materials, namely those of power-law type. While some of our results could be obtained by specializing the work of Ball (1982), it is instructive here to provide a direct *ad hoc* treatment.

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In Section 2, we formulate the basic boundary-value problem that arises when a solid sphere, composed of an incompressible isotropic elastic material, is subjected to a prescribed uniform radial tensile dead-load p_0 on its boundary. One solution to this problem, for all values of p_0, corresponds to a trivial homogeneous state in which the sphere remains undeformed while stressed. However, for sufficiently large values of p_0, one has in addition other possible radially symmetric configurations involving an internal traction-free spherical cavity. Such solutions have been shown by Ball (1982) to bifurcate from the homogeneous solution at a critical value of p_0, say p_{cr}, at which the homogeneous solution becomes unstable. The possibility for these bifurcated solutions to exist depends on the constitutive law for the material under consideration.

In Section 3, attention is confined to a particular class of homogeneous isotropic incompressible elastic materials, namely those of power-law type. Such nonlinearly elastic materials were first introduced by Ogden (1972) and have been employed in a wide variety of problems since then (see e.g. Ogden, 1982, 1984). An extensive discussion of the properties of this class of materials has been provided recently by Zee and Sternberg (1983). Our interest here is in examining the dependence of the critical loads at which cavitation occurs on the constitutive parameter n appearing in the definition of this class of materials [see eqn (47)]. It is found that as the hardening parameter n increases, the critical load p_{cr} at which bifurcation takes place also increases. The stress distribution in the sphere is also described. An interesting feature concerning the principal stresses immediately after cavitation is the presence of a boundary layer near the cavity wall.

2. BIFURCATION PROBLEM FOR A SPHERE: FORMULATION AND SOLUTION

2.1. *Formulation*

We are concerned here with a sphere composed of a homogeneous incompressible isotropic elastic material. Let the undeformed solid sphere be denoted by

$$D_0 = \{(r, \theta, \phi) \mid 0 \leqslant r < b, 0 < \theta \leqslant 2\pi, 0 \leqslant \phi \leqslant \pi\}.$$

The sphere is subjected to a prescribed uniform radial tensile dead-load of magnitude p_0 on its boundary $r = b$. The resulting deformation is a one-to-one mapping which takes the point with spherical polar coordinates (r, θ, ϕ) in the undeformed region D_0 to the point (R, Θ, Φ) in the deformed region D. We assume that the deformation is radially symmetric so that

$$R = R(r) > 0, \quad 0 < r < b; \quad R(0+) \geqslant 0, \quad \Theta = \theta, \quad \Phi = \phi, \quad \text{on } D_0 \tag{1}$$

where $R(r)$ is to be determined.

The spherical polar components of the deformation gradient tensor $\underset{\sim}{F}$ associated with (1) are given by

$$\underset{\sim}{F} = \operatorname{diag}(\dot{R}(r), R(r)/r, R(r)/r) \tag{2}$$

where the dot denotes differentiation with respect to the argument. The principal stretches associated with the radially symmetric deformation (1) are

$$\lambda_r = \dot{R}(r), \quad \lambda_\theta = \lambda_\phi = \frac{R(r)}{r}. \tag{3}$$

Incompressibility then requires that the Jacobian determinant $J = \operatorname{Det} \underset{\sim}{F} = 1$, which upon integration yields

$$R(r) = (r^3+c^3)^{1/3} \tag{4}$$

where $c \geqslant 0$ is a constant to be determined. If it is found that $c = 0$, (4) implies that the body remains a solid sphere in the current configuration. On the other hand, if c is found to be greater than zero, then $R(0+) = c > 0$ and so there is a cavity of radius c centered at the origin in the current configuration. In this event, the cavity surface is assumed to be traction-free.

The strain-energy density per unit undeformed volume for a homogeneous isotropic incompressible elastic material is denoted by

$$W = W(\lambda_1, \lambda_2, \lambda_3) \tag{5}$$

where λ_i ($i = 1, 2, 3$) are the principal stretches. The function W is invariant with respect to interchange of the λ_i and is taken to satisfy the normalization condition $W(1,1,1) = 0$. In the sequel, we proceed formally and assume that W possesses sufficient regularity properties to permit the subsequent analysis.

The principal components of the Cauchy stress tensor $\underset{\sim}{\tau}$ are given by

$$\tau_{ii} = \lambda_i \frac{\partial W}{\partial \lambda_i} - p \quad \text{(no sum on } i\text{)} \tag{6}$$

where p is the hydrostatic pressure associated with the incompressibility constraint $\lambda_1\lambda_2\lambda_3 = 1$. For the radially symmetric deformation with principal stretches given by (3), the principal stress components are

$$\left.\begin{aligned} \tau_{RR}(r) &= v^{-2}W_1(v^{-2}, v, v) - p(r) \\ \tau_{\Theta\Theta}(r) &= \tau_{\Phi\Phi}(r) = vW_2(v^{-2}, v, v) - p(r) \end{aligned}\right\} \tag{7}$$

where, following Ball (1982), we have introduced the notation

$$v = v(r) = \frac{R}{r} = \left(1+\frac{c^3}{r^3}\right)^{1/3}. \tag{8}$$

Notice that in (7) we consider $\tau(r)$ rather than the more conventional $\tau(R)$. The subscript notation on W in (7) denotes differentiation with respect to the appropriate argument. In (7) we have also used $W_2(v^{-2}, v, v) = W_3(v^{-2}, v, v)$, which follows from the invariance of W with respect to interchange of its three arguments.

The dead-load boundary condition now requires that

$$\tau_{RR}(b) = p_0\left[\frac{b}{R(b)}\right]^2 = p_0[v(b)]^{-2} \tag{9}$$

where the constant $p_0 > 0$ is prescribed. We note that the boundary conditions of vanishing shear tractions are satisfied identically. In addition if $c > 0$, then the condition for a traction-free cavity surface

$$\tau_{RR}(0) = 0 \tag{10}$$

must also hold.

In the absence of body forces, the sphere will be in equilibrium provided that div $\underset{\sim}{\tau} = \underset{\sim}{0}$, which will hold provided that

$$\frac{\partial \tau_{RR}}{\partial r}+2\frac{\dot{R}}{R}[\tau_{RR}-\tau_{\Theta\Theta}]=0 \tag{11}$$

holds throughout the sphere.

Thus, the problem to be solved is the following: *for a prescribed value of the dead-load traction $p_0 > 0$, we seek a pressure field $p(r)$ and a constant $c \geqslant 0$ such that* (11) *and* (9) *are satisfied where τ_{RR}, $\tau_{\Theta\Theta}$, $\tau_{\Phi\Phi}$ are given by* (7) *and* (8). *In addition if $c > 0$, then* (10) *must also be satisfied.*

2.2. *Solutions*

It may be readily shown that one solution to the foregoing problem, for all values of p_0, is

$$p(r) = W_1(1,1,1)-p_0, \quad c = 0. \tag{12}$$

This corresponds to the trivial homogeneous state of deformation

$$R(r) = r \tag{13}$$

with corresponding stresses $\tau_{RR} = \tau_{\Theta\Theta} = \tau_{\Phi\Phi} = p_0$.

Next we describe solutions for which $c > 0$, corresponding to the presence of a traction-free cavity at the origin. For this purpose, we adopt an approach developed by Horgan and Pence (1989a) and rewrite the differential equation (11) in the form

$$\frac{\partial}{\partial r}[v^{-2}W_1(v^{-2},v,v)-p(r)]+\frac{2v^{-4}}{r}[v^{-1}W_1(v^{-2},v,v)-v^2W_2(v^{-2},v,v)] = 0, \quad \text{on } 0<r<b \tag{14}$$

where we have used (7), (8). On integration of (14), we have

$$p(r)-p(0) = v^{-2}(r)W_1(v^{-2},v,v)+2J(r), \quad 0<r<b \tag{15}$$

where

$$J(r) = \int_0^r \{v^{-5}(s)W_1[v^{-2}(s),v(s),v(s)]-v^{-2}(s)W_2[v^{-2}(s),v(s),v(s)]\}\frac{\mathrm{d}s}{s}, \quad 0<r<b. \tag{16}$$

On substitution into (7) we obtain

$$\tau_{RR}(r) = -p(0)-2J(r), \quad 0<r<b. \tag{17}$$

The traction-free cavity surface condition (10), together with (17) and $J(0) = 0$, now yields

$$p(0) = 0. \tag{18}$$

Finally the boundary condition (9) at $r = b$ is satisfied if

$$-2J(b) = p_0[v(b)]^{-2}. \tag{19}$$

The condition (19) may be written in a compact fashion on utilizing the change of variables $s \to v$ in the integral (16). From (8) it is seen that this change of variable is one-to-one and invertible if and only if $c > 0$. Introducing the function

$$\hat{W}(x) = W(x^{-2}, x, x) \tag{20}$$

and adopting the notation

$$\hat{W}_1(x) = \frac{\mathrm{d}}{\mathrm{d}x}\hat{W}(x), \tag{21}$$

(19) may be written as

$$p_0 = \left(1+\frac{c^3}{b^3}\right)^{2/3} \int_{(1+(c^3/b^3))^{1/3}}^{\infty} \frac{\hat{W}_1(v)}{(v^3-1)}\,\mathrm{d}v, \quad c > 0. \tag{22}$$

Equation (22) was first established by Ball (1982) for the n-dimensional version of the problem described here [see eqn (5.18) of Ball (1982)]. Thus, for a given dead-load p_0, solutions involving a traction-free internal cavity of radius c exist provided that c is a positive root of (22). The associated pressure field is given by

$$p(r) = v^{-2}(r)W_1(v^{-2}, v, v) + 2J(r), \quad 0 < r < b. \tag{23}$$

The *critical load* p_{cr} at which an internal cavity may be initiated is found by formally letting $c \to 0+$ in (22), and so

$$p_{cr} = \int_1^{\infty} \frac{\hat{W}_1(v)}{(v^3-1)}\,\mathrm{d}v. \tag{24}$$

This result was first established by Ball (1982) in n-dimensions [see eqn (5.22) of Ball (1982)].

In summary then, we have seen that for all values of the applied dead-load traction p_0, one obtains the trivial solution (12) corresponding to the homogeneous state of deformation (13). Moreover, if positive roots c of (22) exist, then one obtains the *additional* solutions involving a traction-free internal cavity described above. Such solutions have been shown by Ball (1982) to bifurcate from the trivial solution at the critical value p_{cr} at which the trivial solution becomes unstable.

2.3. *The critical load*

Since the integral in (24) is improper, p_{cr} may or may not be finite, and so cavitation may or may not take place. As regards the lower limit in (24), it is easily shown that

$$\frac{\mathrm{d}\hat{W}(1)}{\mathrm{d}v} = 0, \quad \frac{\mathrm{d}^2\hat{W}(1)}{\mathrm{d}v^2} = 12\mu \tag{25}$$

where μ denotes the shear modulus for infinitesimal deformations of the material. Thus by l'Hôpital's rule, the limit of the integrand in (24) is finite as $v \to 1$. An analogous issue was discussed by Horgan and Pence (1989b) in the context of a composite sphere under tensile dead-loading on its boundary. Consequently the question of whether or not p_{cr} is finite depends on the behavior of $\hat{W}(v)$ for *large* values of stretch v. Sufficient conditions to guarantee that p_{cr} be finite were given by Ball (1982) for both incompressible and compressible materials. Here we provide an *ad hoc* treatment of this issue. Suppose, for example, that the strain-energy density per unit undeformed volume for a homogeneous incompressible isotropic elastic material can be written in the polynomial form

$$\hat{W}(v) = a_0 + a_1 v + a_2 v^2 + \cdots + a_n v^n \quad (n > 1) \tag{26}$$

so that

$$\hat{W}_1(v) = a_1 + 2a_2 v + \cdots + n a_n v^{n-1}. \tag{27}$$

From (24), (27) we see that p_{cr} will be finite if

$$v^{n-4} < v^{-1} \quad \text{for large } v. \tag{28}$$

Thus if

$$n < 3 \tag{29}$$

the value of p_{cr} given by (24) will be finite.

We now consider some specific examples.

Example 1. *The neo-Hookean material.* The strain-energy density function for this material is given by

$$W(\lambda_1, \lambda_2, \lambda_3) = \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3), \quad \lambda_1 \lambda_2 \lambda_3 = 1 \tag{30}$$

where $\lambda_1, \lambda_2, \lambda_3$, are the principal stretches, and $\mu > 0$ is the shear modulus for infinitesimal deformations. By virtue of (3), (8) and (20) we thus have

$$\hat{W}(v) = \frac{\mu}{2}(v^{-4} + 2v^2 - 3). \tag{31}$$

Therefore

$$\hat{W}(v) \to \mu v^2 \quad \text{for large } v. \tag{32}$$

Thus comparing with (26), we get $n = 2$ and so by (29), the critical load p_{cr} is finite. In fact Ball (1982) has shown that

$$p_{cr} = 5\mu/2. \tag{33}$$

(See also Section 3 of the present paper.)

Example 2. *The Mooney–Rivlin material.* The strain-energy density function for this material is

$$W(\lambda_1, \lambda_2, \lambda_3) = \frac{\mu_1}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + \frac{\mu_2}{2}(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 - 3), \quad \lambda_1\lambda_2\lambda_3 = 1 \tag{34}$$

where λ_1, λ_2 and λ_3 are the principal stretches, and μ_1, μ_2 are positive constants. By virtue of (3), (8), and (20) we thus have

$$\hat{W}(v) = \frac{\mu_1}{2}(v^{-4} + 2v^2 - 3) + \frac{\mu_2}{2}(v^4 + 2v^{-2} - 3). \tag{35}$$

Therefore

$$\hat{W}(v) \to \frac{\mu_2 v^4}{2} \quad \text{for large } v. \tag{36}$$

Thus comparing with (26), we see that $n = 4$, and so by (29), the critical load p_{cr} is *not* finite. Of course, it is well known that the Mooney–Rivlin model is not a very accurate constitutive model for large stretches (see, for example, Ogden, 1984, pp. 492–493 for a discussion of biaxial deformation of a rectangular sheet).

Example 3. *The Rivlin–Saunders material.* Experimental work of Rivlin and Saunders (1951) suggests consideration of a strain-energy density function of the form

$$W(\lambda_1,\lambda_2,\lambda_3) = \frac{\mu_1}{2}(\lambda_1^2+\lambda_2^2+\lambda_3^2-3)+f(\lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_3^2\lambda_1^2-3), \quad \lambda_1\lambda_2\lambda_3 = 1 \quad (37)$$

where f is, as yet, an unspecified function, with $f(0) = 0$ and μ_1 is a positive constant. By virtue of (3), (8) and (20) we thus have

$$\hat{W}(v) = \frac{\mu_1}{2}(v^{-4}+2v^2-3)+f(v^4+2v^{-2}-3). \quad (38)$$

In what follows, we discuss two special forms of (38).

$$\text{(i)} \quad \hat{W}(v) = \frac{\mu_1}{2}(v^{-4}+2v^2-3)+\frac{\mu_2}{2}(v^4+2v^{-2}-3)^{\alpha}, \quad \alpha > 0, \quad \mu_2 > 0. \quad (39)$$

Clearly the special case $\alpha = 1$ corresponds to the Mooney–Rivlin material (34) considered in Example 2 above. We see that if $4\alpha > 2$ $(\alpha > 1/2)$,

$$\hat{W}(v) \to \frac{\mu_2 v^{4\alpha}}{2} \quad \text{for large } v. \quad (40)$$

On comparing with (26), we have $n = 4\alpha$, and so by (29) to ensure that p_{cr} is finite, we require that $4\alpha < 3$, i.e. $\alpha < 3/4$, and so p_{cr} is finite for the material (39) if

$$\tfrac{1}{2} < \alpha < \tfrac{3}{4}. \quad (41)$$

For $\alpha \leqslant 1/2$,

$$\hat{W}(v) \to \mu_1 v^2 \quad \text{for large } v \quad (42)$$

and so p_{cr} is again finite on comparing with (26) and (29). In summary then, for the material (39), p_{cr} is finite if

$$0 < \alpha < \tfrac{3}{4}. \quad (43)$$

We remark that the commonly used version of (39) with $\alpha = 2$ does *not* yield a finite value of p_{cr}. It is of interest to observe here that Simmonds (1989) has recently shown that a circular rubber-like plate composed of the material (39) suffers a *finite* deflection under a concentrated vertical load at its center only if $\alpha > 1$. For a membrane it is shown by Fulton and Simmonds (1986) that the corresponding result holds only if $\alpha > 2$. See also the discussion on pp. 281 and 282 of the book by Libai and Simmonds (1988).

(ii) Another special form of (37) has been considered by Gent and Thomas (1958), in which f is taken to be the logarithm function. Thus we have

$$\hat{W}(v) = \frac{\mu_1}{2}(v^{-4}+2v^2-3)+\frac{\mu_2}{2}\ln(v^4+2v^{-2}-2), \quad \mu_2 > 0 \quad (44)$$

so that

$$\hat{W}_1(v) = \frac{\mu_1}{2}(-4v^{-5}+4v)+\frac{\mu_2}{2}\frac{4v^3-4v^{-3}}{v^4+2v^{-2}-2}. \tag{45}$$

Thus

$$\hat{W}_1(v) \to 2\mu_1 v \quad \text{for large } v \tag{46}$$

and so from (24) we see that the critical load p_{cr} is finite for the material (44).

3. SOLUTIONS FOR A CLASS OF INCOMPRESSIBLE ELASTIC MATERIALS

3.1. *A class of incompressible elastic materials*

We now consider a particular constitutive law, namely that of power-law type, and provide an explicit solution for the bifurcation problem discussed generally in Section 2. Thus consider

$$W(\lambda_1,\lambda_2,\lambda_3) = \frac{\mu}{2n}(\lambda_1^{2n}+\lambda_2^{2n}+\lambda_3^{2n}-3), \quad \lambda_3 = (\lambda_1\lambda_2)^{-1}, \quad \mu > 0, \quad n > 0 \tag{47}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches, and the constants μ, n are constitutive parameters. Constitutive models of the form (47) were first introduced by Ogden (1972) and have been widely investigated since then (see e.g. Ogden, 1982, 1984). The constant μ in (47) is the shear modulus for infinitesimal deformations and n is the hardening exponent. The special case when $n = 1$ in (47) corresponds to the neo-Hookean material.

We recall from Section 2 that the critical load p_{cr} is given by (24), i.e.

$$p_{cr} = \int_1^{\infty} \frac{\hat{W}_1(v)}{(v^3-1)}\,dv \tag{48}$$

where the notation (20), (21) is used. Expressed in the notation of (20), the strain-energy density (47) can be written in polynomial form as

$$\hat{W}(v) = \frac{\mu}{2n}(v^{-4n}+2v^{2n}-3), \quad \mu > 0, \quad n > 0. \tag{49}$$

To ensure the existence of p_{cr}, we recall from (29) that $2n$ should be less than 3, i.e.

$$n < \tfrac{3}{2}. \tag{50}$$

It is of interest to observe that a restriction similar to (50) also arises in the work of Carroll (1987) concerned with the problem of inflation of a hollow sphere composed of the material (47).

The response of the material described by (47) to certain basic pure homogeneous deformations will now be discussed. A recent investigation of these issues was carried out by Zee and Sternberg (1983), and we now summarize their results which are relevant to our problem here. The pure homogeneous deformations considered are as follows:

$$\left.\begin{array}{l}
\text{(i) } \textit{uniaxial stress} \\
\quad \tau_{11} = \tau_{22} = 0,\ \tau_{33}(\lambda) = \mu(\lambda^{2n}-\lambda^{-n}),\ \lambda_3 = \lambda,\ \lambda_1 = \lambda_2 = \lambda^{-1/2} \\
\text{(ii) } \textit{equibiaxial stress} \\
\quad \tau_{33} = 0,\ \tau_{22}(\lambda) = \mu(\lambda^{2n}-\lambda^{-4n}),\ \lambda_1 = \lambda_2 = \lambda,\ \lambda_3 = \lambda^{-2} \\
\text{(iii) } \textit{pure shear} \\
\quad \tau_{22} = 0,\ \tau_{11}(\lambda) = \mu(\lambda^{2n}-\lambda^{-2n}),\ \lambda_1 = \lambda_2^{-1} = \lambda,\ \lambda_3 = 1.
\end{array}\right\} \tag{51}$$

The normal stresses $\tau_{11}(\lambda)$, $\tau_{22}(\lambda)$, as well as $\tau_{33}(\lambda)$, for each of the pure homogeneous

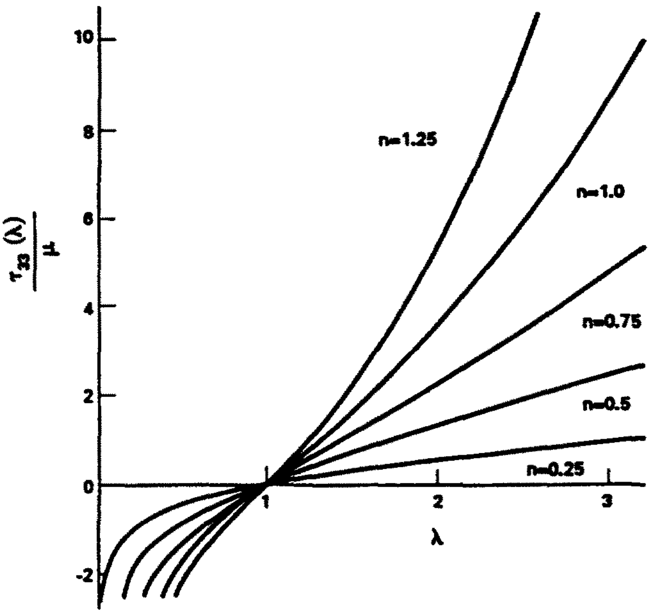

Fig. 1. Behavior of the power-law material under uniaxial stress.

deformations (51), are monotonic increasing functions of λ for $0 < \lambda < \infty$. The stress–stretch relation (51) appropriate to (i) (uniaxial stress) is plotted in Fig. 1 for the values of the exponent n given by $n = 5/4$, 1, 3/4, 1/2, and 1/4 (cf. Fig. 3 of Zee and Sternberg, 1983). Note that the material hardens as n increases. The graphs of $\tau_{22}(\lambda)$ and $\tau_{11}(\lambda)$, corresponding to the cases (ii) and (iii), are qualitatively similar to Fig. 1.

It is of interest to remark on the character of the system of governing partial differential equations, namely the displacement equations of equilibrium

$$C_{ijkl}(\underset{\sim}{F})u_{k,lj} - p_{,j}F_{ji}^{-1} = 0; \quad J = \det \underset{\sim}{F} = 1 \tag{52}$$

where $C_{ijkl}(\underset{\sim}{F})$ are the components of the fourth-order tensor defined by

$$C_{ijkl}(\underset{\sim}{F}) = C_{klij}(\underset{\sim}{F}) = \frac{\partial^2 W(\underset{\sim}{F})}{\partial F_{ij}\,\partial F_{kl}}. \tag{53}$$

Necessary and sufficient conditions for *ellipticity* of the system of equations (52), (53) have been obtained by Zee and Sternberg (1983). For the special case of the material (47), these conditions are particularly simple. Thus from the results of Zee and Sternberg (1983, p. 85), ellipticity holds for the material (47) *at all deformations* if

$$n \geqslant \tfrac{1}{2}. \tag{54}$$

In what follows, we assume that (54) holds, and so recalling (50), we thus have

$$\tfrac{1}{2} \leqslant n < \tfrac{3}{2}. \tag{55}$$

3.2. *Cavitation solutions*

Consider a quasi-static loading process in which the solid sphere is subjected to a dead-load p_0 that increases slowly from zero. Cavity formation and growth is described by the relationship $p_0 = p_0(c)$ given in (22).

For the material described by (47) [recalling the notation (3), (8)] we have $\lambda_1 = v^{-2}$, $\lambda_2 = \lambda_3 = v$ and so the first derivative with respect to λ_2 is 0, i.e. $W_2(v^{-2}, v, v) = 0$, and the first derivative with respect to λ_1 is given by

$$W_1(v^{-2}, v, v) = \mu(v^{-4n+2} - v^{2n+2}), \quad \tfrac{1}{2} \leqslant n < \tfrac{3}{2}. \tag{56}$$

On using the notation (20), (21) we obtain from (49)

$$\hat{W}_1(v) = 2\mu(v^{2n-1} - v^{-4n-1}), \quad \tfrac{1}{2} \leqslant n < \tfrac{3}{2} \tag{57}$$

and so

$$\frac{\hat{W}_1(v)}{v^3-1} = 2\mu \frac{v^{2n-1} - v^{-4n-1}}{v^3-1}, \quad \tfrac{1}{2} \leqslant n < \tfrac{3}{2}. \tag{58}$$

When the relationship (22) between the applied pressure p_0 and deformed cavity radius c is specialized to the particular strain-energy function (47) [and (58) is used], one obtains

$$p_0 = p_0(c) = 2\mu\left(1 + \frac{c^3}{b^3}\right)^{2/3} \int_{(1+(c^3/b^3))^{1/3}}^{\infty} \frac{v^{2n-1} - v^{-4n-1}}{v^3-1}\, dv, \quad \tfrac{1}{2} \leqslant n < \tfrac{3}{2}. \tag{59}$$

Before proceeding with an analysis of the relationship (59), it is convenient to record here corresponding expressions for the stresses subsequent to cavitation given by (7). On using (17), (18), (20), (21) we find

$$\tau_{RR}(r) = \int_{(1+(c^3/r^3))^{1/3}}^{\infty} \frac{\hat{W}_1(v)}{v^3-1}\, dv \tag{60}$$

while from (7) we obtain

$$\tau_{\Theta\Theta} = \tau_{\Phi\Phi} = vW_2(v^{-2}, v, v) - v^{-2}W_1(v^{-2}, v, v) + \tau_{RR}(r). \tag{61}$$

On using $W_2(v^{-2}, v, v) = 0$, (56) and (58) we obtain

$$\tau_{RR}(r) = 2\mu \int_{(1+(c^3/r^3))^{1/3}}^{\infty} \frac{v^{2n-1} - v^{-4n-1}}{v^3-1}\, dv, \quad \tfrac{1}{2} \leqslant n < \tfrac{3}{2} \tag{62}$$

and

$$\tau_{\Theta\Theta} = \tau_{\Phi\Phi} = \tau_{RR}(r) - \mu[v^{-4n}(r) - v^{2n}(r)] \tag{63}$$

where we recall from (8) that $v(r) = (1 + (c^3/r^3))^{1/3}$.

We confine attention to the range of values of n in (55), namely $1/2 \leqslant n < 3/2$. For specific values of n in this range, namely $n = 1/2, 3/4, 1, 5/4$, the integrals in (59) and (62) may be evaluated explicitly. The relevant integrals can be evaluated by using results of Ryshik and Gradstein (1963). We assemble these integrals in the Appendix. The corresponding expressions occurring in (59) then become

$$n = \tfrac{1}{2}: \quad p_0 = \mu. \tag{64}$$

$$n = \frac{3}{4}: \quad p_0 = 2\mu\left(1+\frac{c^3}{b^3}\right)^{2/3}\left[\frac{1}{3}\ln\frac{1+(1+c^3/b^3)^{1/2}}{1+c^3/b^3}+\frac{1}{3}\left(1+\frac{c^3}{b^3}\right)^{-1} + \frac{1}{3}\ln\frac{c^3/b^3}{(1+c^3/b^3)^{1/2}-1}\right] \quad (65)$$

$n = 1$: (neo-Hookean material)

$$p_0 = 2\mu\left(1+\frac{c^3}{b^3}\right)^{-2/3}\left[\frac{5+4c^3/b^3}{4}\right] \quad (66)$$

$n = \frac{5}{4}$:

$$\begin{aligned} p_0 = 2\mu\left(1+\frac{c^3}{b^3}\right)^{2/3}&\left\{\frac{4}{3\sqrt{3}}\pi - \frac{1}{3}\ln\left[\frac{(1+c^3/b^3)^{1/6}-1}{(1+c^3/b^3)^{1/3}-1}\right]\right. \\ &+\frac{1}{3}\ln\left[1+\left(1+\frac{c^3}{b^3}\right)^{1/6}\right]-\frac{1}{6}\ln\left[1-\left(1+\frac{c^3}{b^3}\right)^{1/6}+\left(1+\frac{c^3}{b^3}\right)^{1/3}\right] \\ &+\frac{1}{6}\ln\left[1+\left(1+\frac{c^3}{b^3}\right)^{1/6}+\left(1+\frac{c^3}{b^3}\right)^{1/3}\right]-\frac{1}{\sqrt{3}}\arctan\frac{2(1+c^3/b^3)^{1/6}-1}{\sqrt{3}} \\ &-\frac{1}{6}\ln\left[1+\left(1+\frac{c^3}{b^3}\right)^{1/3}+\left(1+\frac{c^3}{b^3}\right)^{2/3}\right]-\frac{1}{\sqrt{3}}\arctan\frac{2(1+c^3/b^3)^{1/6}+1}{\sqrt{3}} \\ &\left.+\frac{1}{5}\left(1+\frac{c^3}{b^3}\right)^{-5/3}+\frac{1}{2}\left(1+\frac{c^3}{b^3}\right)^{-2/3}-\frac{1}{\sqrt{3}}\arctan\frac{\sqrt{3}(1+c^3/b^3)^{1/3}}{2+(1+c^3/b^3)^{1/3}}\right\}. \end{aligned} \quad (67)$$

Equations (64)–(67) provide a relationship $p_0 = p_0(c)$ between the dimensionless applied dead-load p_0/μ and the dimensionless cavity radius c/b. The critical load p_{cr} is the value at which the curve $p_0 = p_0(c)$ bifurcates from the straight line $c = 0$ corresponding to the trivial homogeneous solution. On letting $c \to 0+$ in (64)–(67) and applying l'Hôpital's rule where appropriate, the critical load p_{cr} is tabulated below.

n:	1/2	3/4	1	5/4
p_{cr}:	μ	1.5909μ	2.5μ	4.7426μ

As one might expect, the values of p_{cr} increase as the hardening parameter n increases. The graphs of $p_0(c)$ according to (64)–(67) are shown in Fig. 2. From Fig. 2 [and (64)], it is clear that the case $n = 1/2$ is special. We recall from (54) that this is the limiting value of n for which ellipticity holds.

The corresponding principal stresses, given by (62), (63), are

$n = \frac{1}{2}$:

$$\tau_{RR}(r) = \mu\left(1+\frac{c^3}{r^3}\right)^{-2/3} \quad (68)$$

$$\tau_{\Theta\Theta}(r) = \tau_{\Phi\Phi}(r) = \mu\left(1+\frac{c^3}{r^3}\right)^{1/3} \quad (69)$$

$n = \frac{3}{4}$:

$$\tau_{RR}(r) = 2\mu\left[\frac{1}{3}\ln\frac{1+(1+c^3/r^3)^{1/2}}{1+c^3/r^3}+\frac{1}{3}\left(1+\frac{c^3}{r^3}\right)^{-1}+\frac{1}{3}\ln\frac{c^3/r^3}{(1+c^3/r^3)^{1/2}-1}\right] \quad (70)$$

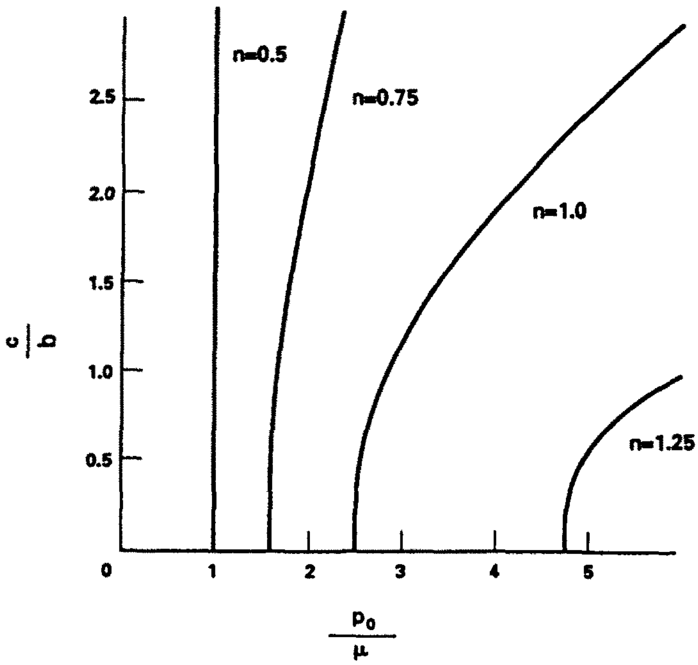

Fig. 2. Variation of the deformed cavity radius c with applied dead load p_0 for a power-law material with strain energy density given by (47).

$$\tau_{\Theta\Theta}(r) = \tau_{\Phi\Phi}(r) = \mu\left[\frac{2}{3}\ln\frac{1+(1+c^3/r^3)^{1/2}}{1+c^3/r^3} - \frac{1}{3}\left(1+\frac{c^3}{r^3}\right)^{-1} + \frac{2}{3}\ln\frac{c^3/r^3}{(1+c^3/r^3)^{1/2}-1} + \left(1+\frac{c^3}{r^3}\right)^{1/2}\right] \tag{71}$$

$n = 1$:

$$\tau_{RR}(r) = 2\mu\left[\left(1+\frac{c^3}{r^3}\right)^{-1/3} + \frac{1}{4}\left(1+\frac{c^3}{r^3}\right)^{-4/3}\right] \tag{72}$$

$$\tau_{\Theta\Theta}(r) = \tau_{\Phi\Phi}(r) = \tau_{RR}(r) - \mu\left[\left(1+\frac{c^3}{r^3}\right)^{-4/3} - \left(1+\frac{c^3}{r^3}\right)^{2/3}\right] \tag{73}$$

$n = \frac{5}{4}$:

$$\begin{aligned}\tau_{RR}(r) = 2\mu\Bigg\{&\frac{4}{3\sqrt{3}}\pi - \frac{1}{3}\ln\left[\frac{(1+c^3/r^3)^{1/6}-1}{(1+c^3/r^3)^{1/3}-1}\right] \\ &+ \frac{1}{3}\ln\left[1+\left(1+\frac{c^3}{r^3}\right)^{1/6}\right] - \frac{1}{6}\ln\left[1-\left(1+\frac{c^3}{r^3}\right)^{1/6}+\left(1+\frac{c^3}{r^3}\right)^{1/3}\right] \\ &+ \frac{1}{6}\ln\left[1+\left(1+\frac{c^3}{r^3}\right)^{1/6}+\left(1+\frac{c^3}{r^3}\right)^{1/3}\right] - \frac{1}{\sqrt{3}}\arctan\frac{2(1+c^3/r^3)^{1/6}-1}{\sqrt{3}} \\ &- \frac{1}{6}\ln\left[1+\left(1+\frac{c^3}{r^3}\right)^{1/3}+\left(1+\frac{c^3}{r^3}\right)^{2/3}\right] - \frac{1}{\sqrt{3}}\arctan\frac{2(1+c^3/r^3)^{1/6}+1}{\sqrt{3}} \\ &+ \frac{1}{5}\left(1+\frac{c^3}{r^3}\right)^{-5/3} + \frac{1}{2}\left(1+\frac{c^3}{r^3}\right)^{-2/3} - \frac{1}{\sqrt{3}}\arctan\frac{\sqrt{3}(1+c^3/r^3)^{1/3}}{2+(1+c^3/r^3)^{1/3}}\Bigg\}\end{aligned} \tag{74}$$

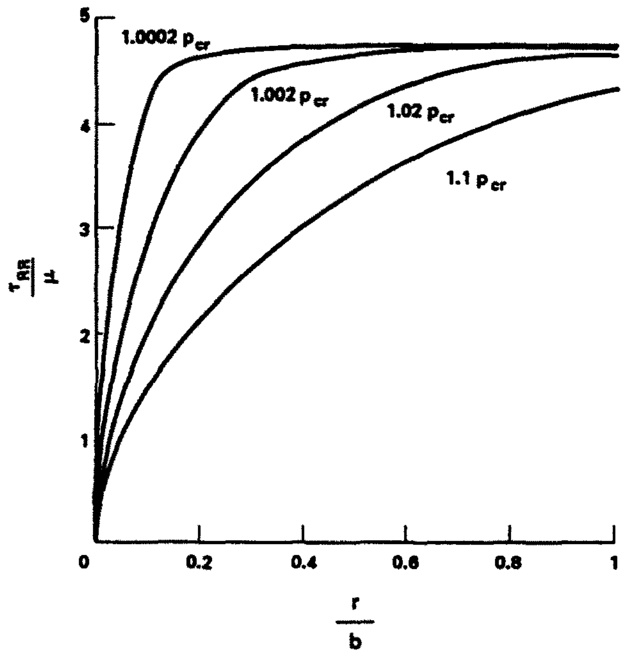

Fig. 3. Variation of the radial stress $\tau_{RR}(r)$ with undeformed radius r subsequent to cavitation for a power-law material (47) with $n = 5/4$.

$$\tau_{\Theta\Theta}(r) = \tau_{\Phi\Phi}(r) = \tau_{RR}(r) - \mu\left[\left(1+\frac{c^3}{r^3}\right)^{-5/3} - \left(1+\frac{c^3}{r^3}\right)^{5/6}\right]. \tag{75}$$

Graphs of $\tau_{RR}(r)$, $\tau_{\Theta\Theta}(r)$ and $\tau_{\Phi\Phi}(r)$ for $n = 5/4$, are shown in Figs 3, 4. (The corresponding graphs for $n = 1, 3/4$ are qualitatively similar.) An interesting feature concerning these stresses immediately after cavitation is the presence of a boundary layer near the

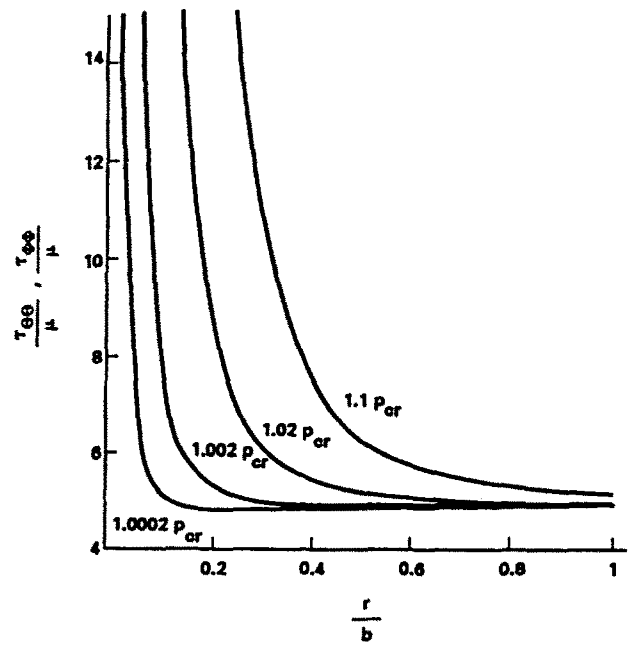

Fig. 4. Variation of the stresses $\tau_{\Theta\Theta}(r)$, $\tau_{\Phi\Phi}(r)$ with undeformed radius r subsequent to cavitation for a power-law material (47) with $n = 5/4$.

cavity wall. To see this, we have plotted the stresses in Figs 3, 4 for applied dead loads p_0 slightly larger than p_{cr}. A similar boundary-layer phenomenon was observed by Horgan and Pence (1989b,c) for the problem of tensile dead-loading of a composite sphere composed of two neo-Hookean materials. The implications of these boundary-layers in the stresses are currently under investigation.

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APPENDIX: VERIFICATION OF (64)–(67) AND (68), (70), (72), (74)

Here we present the details of the derivation of eqns (64)–(67) and (68), (70), (72), (74). We first treat the indefinite integral which is needed to evaluate both (59) and (62). (Constants of integration will not be written down.)

$$I = \int \frac{v^{2n-1} - v^{-4n-1}}{v^3 - 1}\,\mathrm{d}v, \quad \tfrac{1}{2} \leqslant n < \tfrac{3}{2}. \tag{A1}$$

It is convenient to record here the values of $2n-1$ and $-4n-1$ corresponding to $n = 1/2$, $3/4$, 1, $5/4$, respectively. The integral (A1) will be decomposed into the two parts involving these exponents.

n:	1/2	3/4	1	5/4
$2n-1$:	0	1/2	1	3/2
$-4n-1$:	−3	−4	−5	−6

(i) *Evaluation of* I *for* n = 1/2

When $n = 1/2$,

$$I = \int \frac{1 - v^{-3}}{v^3 - 1}\,\mathrm{d}v = \int \frac{v^{-3}(v^3 - 1)}{v^3 - 1}\,\mathrm{d}v = \int v^{-3}\,\mathrm{d}v = -\frac{1}{2v^2}. \tag{A2}$$

On using (A2), the definite integrals in (59), (62) are immediately evaluated to yield the desired expressions (64), (68).

(ii) *Evaluation of* I *for* n = 3/4

First, we record here the indefinite integrals (2.128) of Ryshik and Gradstein (1963).

$$\int \frac{\mathrm{d}v}{v^k z_3^l} = -\frac{1}{(k-1)a v^{k-1} z_3^{l-1}} - \frac{b(3l+k-4)}{a(k-1)} \int \frac{\mathrm{d}v}{v^{k-3} z_3^l}, \quad k \neq 1 \tag{A3}$$

where $z_3 = a + bv^3$, $a \neq 0$, b and $l > 0$ are constants.

When $n = 3/4$, from (A1) we see that

$$I = \int \frac{\sqrt{v}}{(v^3 - 1)}\,\mathrm{d}v - \int \frac{\mathrm{d}v}{v^4(v^3 - 1)} = I_1 - I_2. \tag{A4}$$

To evaluate I_2, we use (A3) with $k = 4$, $a = -1$, $b = 1$, $l = 1$, and get

$$I_2 = \int \frac{\mathrm{d}v}{v^4(v^3 - 1)} = \frac{1}{3v^3} - \int \frac{\mathrm{d}v}{v(v^3 - 1)}. \tag{A5}$$

The second integral of (A5) is evaluated as follows:

$$\int \frac{\mathrm{d}v}{v(v^3 - 1)} = \int \frac{v^2\,\mathrm{d}v}{v^3(v^3 - 1)} = \int -\left(\frac{v^2}{v^3}\right)\mathrm{d}v + \int \frac{v^2}{(v^3 - 1)}\,\mathrm{d}v = -\frac{1}{3}\ln v^3 + \frac{1}{3}\ln(v^3 - 1) = -\frac{1}{3}\ln\frac{v^3}{v^3 - 1} \tag{A6}$$

and so, from (A5), we have

$$I_2 = \frac{1}{3v^3} + \frac{1}{3}\ln\frac{v^3}{v^3 - 1}. \tag{A7}$$

In order to evaluate I_1 in (A4) we use a change of variables, i.e. $r = v^{3/2}$, and so

$$I_1 = \int \frac{\sqrt{v}\,\mathrm{d}v}{v^3 - 1} = \int \frac{2/3\,\mathrm{d}r}{r^2 - 1} = \frac{1}{3}\ln\frac{r - 1}{r + 1} = \frac{1}{3}\ln\frac{v^{3/2} - 1}{v^{3/2} + 1}. \tag{A8}$$

Thus on combining (A7) and (A8) in (A4) we obtain an expression for I. The definite integrals in (59) and (62) are then immediately evaluated to yield the desired expressions (65) and (70).

(iii) *Evaluation of* I *for* n = 1

When $n = 1$,

$$I = \int \frac{v - v^{-5}}{v^3 - 1}\,\mathrm{d}v = \int \frac{v^{-5}(v^6 - 1)}{v^3 - 1}\,\mathrm{d}v = \int v^{-5}(v^3 + 1)\,\mathrm{d}v$$

$$= \int (v^{-2} + v^{-5})\,\mathrm{d}v = -\frac{1}{v} - \frac{1}{4v^4}. \tag{A9}$$

On using (A9), the definite integrals in (59), (62) are immediately evaluated to yield the desired expressions (66), (72).

(iv) *Evaluation of* I *for* n = 5/4
When $n = 5/4$,

$$I = \int \frac{v\sqrt{v}\,dv}{v^3-1} - \int \frac{dv}{v^6(v^3-1)} = I_3 - I_4. \tag{A10}$$

To evaluate I_4, we use (A3) with $k = 6$, $a = -1$, $b = 1$, $l = 1$, to get

$$I_4 = \frac{1}{5v^5} + \int \frac{dv}{v^3(v^3-1)}. \tag{A11}$$

The integral in (A11) is evaluated by using (A3) with $k = 3$, $a = -1$, $b = 1$, $l = 1$, to get

$$\int \frac{dv}{v^3(v^3-1)} = \frac{1}{2v^2} + \int \frac{dv}{v^3-1}. \tag{A12}$$

The last integral in (A12) can be evaluated using standard integral tables. For example (2.143) of Ryshik and Gradstein (1963) gives

$$\int \frac{dv}{v^3-1} = -\frac{1}{3}\ln\frac{(1+v+v^2)^{1/2}}{v-1} - \frac{1}{\sqrt{3}}\arctan\frac{\sqrt{3}v}{2+v}. \tag{A13}$$

Thus, on using (A13), (A12), (A11), we obtain

$$I_4 = \frac{1}{5v^5} + \frac{1}{2v^2} - \frac{1}{3}\ln\frac{(1+v+v^2)^{1/2}}{v-1} - \frac{1}{\sqrt{3}}\arctan\frac{\sqrt{3}v}{2+v}. \tag{A14}$$

In order to evaluate I_3, we use a change of variable, i.e. $r = v^{1/2}$, to get

$$I_3 = \int \frac{v\sqrt{v}\,dv}{v^3-1} = \int \frac{r^4\,dr}{r^6-1} = \int \frac{r\,dr}{r^3+1} + \int \frac{r\,dr}{r^3-1}. \tag{A15}$$

By using (2.145.3) and (2.145.7) of Ryshik and Gradstein (1963), we have

$$\begin{aligned} I_3 = &-\frac{1}{6}\ln\frac{(1+\sqrt{v})^2}{1-\sqrt{v}+v} + \frac{1}{\sqrt{3}}\arctan\frac{2\sqrt{v}-1}{\sqrt{3}} \\ &+\frac{1}{6}\ln\frac{(\sqrt{v}-1)^2}{1+\sqrt{v}+v} + \frac{1}{\sqrt{3}}\arctan\frac{2\sqrt{v}+1}{\sqrt{3}}. \end{aligned} \tag{A16}$$

Thus on combining (A14) and (A16), we obtain an expression for I from (A10). The definite integrals in (59) and (62) are then readily evaluated to obtain the desired expressions (67) and (74).